

# Morita's Theory for the Symplectic Groups

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**ABSTRACT.** We construct and study the holomorphic discrete series representation and the principal series representation of the symplectic group  $\mathrm{Sp}(2n, F)$  over a  $p$ -adic field  $F$  as well as a duality between some sub-representations of these two representations. The constructions of these two representations generalize those defined in Morita and Murase's works. Moreover, Morita built a duality for  $\mathrm{SL}(2, F)$  defined by residues. We view the duality constructed here as an algebraic interpretation of Morita's duality in some extent and its generalization to the symplectic groups.

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*Key words and phrases.* symplectic groups,  $p$ -adic Siegel upper half-space, principal series, holomorphic discrete series, Morita's duality, Casselman's intertwining operator.

### Notations

Let  $p$  be a prime,  $F$  be a finite extension of  $\mathbb{Q}_p$ ,  $\mathfrak{o}$  be the ring of integers of  $F$ ,  $\varpi$  be a uniformizer of  $\mathfrak{o}$ ,  $|\cdot|$  be the normalized absolute value, and  $F^{\text{alg}}$  be an algebraic closure of  $F$ . Let  $K$  be an extension of  $F$  with an absolute value extending  $|\cdot|$  such that  $K$  is complete under this absolute value. Because the Hahn-Banach theorem is applied, we assume that  $K$  is spherically complete in §2 and §3.

## 0. Introduction

**Backgrounds.** In [5], Morita and Murase constructed and studied  $p$ -adic holomorphic discrete series representations of  $\text{SL}(2, F)$ . In [8], Schneider introduced the holomorphic discrete series of  $\text{SL}(n+1, F)$  associated to a rational representation of  $\text{GL}(n, F)$ . He showed that, as  $\text{SL}(n+1, F)$ -representation, the space of holomorphic exterior differential  $r$ -forms on the Drinfel'd's space is actually a holomorphic discrete series.

Morita started the systematic study of principal series (parabolic induced representations) of  $\text{SL}(2, F)$  in [6] and [7]. In order to prove the irreducibility conjectures on holomorphic discrete series, Morita later constructed a duality pairing via residues between holomorphic discrete series and principal series ([6]).

**Outline of article.** In the first paragraph, we generalize Morita's constructions to the symplectic groups. In §1.1, we recollect some notions on the symplectic groups. In §1.2, following [6], we give another interpretation of a parabolic induced representation, which is conventionally called a principal series. General results of Féaux de Lacroix on induced representations of  $F$ -Lie groups ([3]) are applied for our purpose. In §1.3, a  $p$ -adic analogue of the Siegel upper half-space, along with an  $F$ -rigid analytic structure, is introduced. The method is similar to the one utilized in the study of Drinfel'd's space in [10]. In §1.4, we introduce the notion of the holomorphic discrete series of  $\text{Sp}(2n, F)$  associated to a  $K$ -rational representation of  $\text{GL}(n, F)$  and prove that the space of rigid analytic exterior differential  $r$ -forms on the Siegel upper half-space can be realized as a holomorphic discrete series.

In the second paragraph, in a purely algebraic way, we construct two invariant closed subspaces of the principal series and the holomorphic discrete series respectively and establish a duality operator between them. We remark that, since the two spaces are of compact type and nuclear  $K$ -Fréchet, respectively, the duality fits into the framework of Schneider and Teitelbaum's theory (cf. [11]).

In the last paragraph, in the case of  $\text{SL}(2, F)$ , we analyze the relations between the duality constructed in the second paragraph and Morita's duality: composing with Casselman's intertwining operator defined by differentiation, Morita's duality coincides with ours up to a constant.

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communicated many important ideas to us. We also want to thank Professor P. Schneider for several comments and advices.

## 1. Symplectic groups and their representations

**1.1. The symplectic group  $\mathrm{Sp}(2n, F)$ .** Let  $n$  be a positive integer and

$$J_n := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

The *symplectic group*  $\mathrm{Sp}(2n, F)$  is the subgroup of  $\mathrm{GL}(2n, F)$  that consists of matrices  $g$  satisfying

$${}^t g J_n g = J_n.$$

If one writes  $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$  ( $A, B, C, D \in \mathrm{M}(n, F)$ ), then  $g \in \mathrm{Sp}(2n, F)$  if and only if either one of the following two conditions holds:

$$(1.1) \quad {}^t A D - {}^t C B = I_n, \quad {}^t A C = {}^t C A, \quad {}^t B D = {}^t D B;$$

$$(1.2) \quad D {}^t A - C {}^t B = I_n, \quad D {}^t C = C {}^t D, \quad B {}^t A = A {}^t B.$$

In the following, we introduce two homogeneous spaces  $\mathcal{P}(n)$  and  $\mathcal{L}(n)$  associated to  $\mathrm{Sp}(2n, F)$ .

Let  $\mathcal{P}(n)$  denote the set of pairs  $(X, Y)$ ,  $X, Y \in \mathrm{M}(n, F)$ , such that

$$X {}^t Y = Y {}^t X, \quad \mathrm{rank}(X \ Y) = n.$$

A right action of  $\mathrm{Sp}(2n, F)$  and a left action of  $\mathrm{GL}(n, F)$  on  $\mathcal{P}(n)$  are defined by

$$(X, Y)g := (XA + YC, XB + YD), \quad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2n, F),$$

$$h(X, Y) := (hX, hY), \quad h \in \mathrm{GL}(n, F),$$

respectively. Let

$$\mathcal{U} := \left\{ \begin{pmatrix} I_n & B \\ 0 & I_n \end{pmatrix} \in \mathrm{Sp}(2n, F) \right\}.$$

We identify the  $\mathrm{Sp}(2n, F)$ -homogeneous space  $\mathcal{U} \backslash \mathrm{Sp}(2n, F)$  with  $\mathcal{P}(n)$  via  $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto (C, D)$  (the inverse map comes from the symplectic Gram-Schmidt process).

Let  $\mathcal{L}(n)$  denote the set of transposed Langrangian subspaces. Define

$$\mathcal{P} := \left\{ \begin{pmatrix} {}^t D^{-1} & B \\ 0 & D \end{pmatrix} \in \mathrm{Sp}(2n, F) \right\}.$$

Then  $\mathcal{L}(n)$  can be identified with the  $\mathrm{Sp}(2n, F)$ -homogeneous space  $\mathcal{P} \backslash \mathrm{Sp}(2n, F)$ . Since  $\mathcal{P}$  is a parabolic subgroup,  $\mathcal{P} \backslash \mathrm{Sp}(2n, F)$  is a smooth projective variety over  $F$ .

In view of  $\mathcal{P} \cong \mathcal{U} \rtimes \mathrm{GL}(n, F)$ , we have a natural  $\mathrm{Sp}(2n, F)$ -equivariant isomorphism  $\mathrm{GL}(n, F) \backslash \mathcal{P}(n) \cong \mathcal{L}(n)$ ; the projection from  $\mathcal{P}(n)$  onto  $\mathcal{L}(n)$  maps  $(X, Y)$  to the transposed Lagrangian subspace spanned by the row vectors of  $(X \ Y)$ .

Finally, we define certain open subsets that define the coordinates on  $\mathrm{Sp}(2n, F)$ ,  $\mathcal{P}(n)$  and  $\mathcal{L}(n)$ .

Let

$$U_0 := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2n, F) : \det(C) \neq 0 \right\}.$$

We have the following unique decomposition in  $\mathrm{Sp}(2n, F)$  for matrices in  $U_0$ :

$$(1.3) \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I_n & AC^{-1} \\ 0 & I_n \end{pmatrix} \begin{pmatrix} {}^t C^{-1} & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} 0 & -I_n \\ I_n & C^{-1}D \end{pmatrix}.$$

$AC^{-1}$  and  $C^{-1}D$  are symmetric ((1.1) and (1.2)). Thus, one may identify  $U_0$  with  $\mathrm{Sym}(n, F) \times \mathrm{GL}(n, F) \times \mathrm{Sym}(n, F)$ .

Let  $\mathcal{U}_0$  be the open subset of  $\mathcal{P}(n)$ :

$$\{(h, hz) : h \in \mathrm{GL}(n, F), z \in \mathrm{Sym}(n, F)\}.$$

Under the identification  $\mathcal{P}(n) \cong \mathrm{U} \backslash \mathrm{Sp}(2n, F)$ , we have  $\mathcal{U}_0 \cong \mathrm{U} \backslash U_0$ .

Furthermore, we identify  $\mathrm{Sym}(n, F)$  with the open subset  $\mathrm{P} \backslash U_0$  of  $\mathcal{L}(n)$ .

To lighten notations, hereafter we let  $G = \mathrm{Sp}(2n, F)$ ,  $G_0 = \mathrm{Sp}(2n, \mathfrak{o})$ ,  $H = \mathrm{GL}(n, F)$ ,  $H_0 = \mathrm{GL}(n, \mathfrak{o})$  and abbreviate  $\mathcal{P}(n)$  and  $\mathcal{L}(n)$  to  $\mathcal{P}$  and  $\mathcal{L}$ , respectively. Moreover, let  $\mathrm{pr}_{\mathcal{P}}^G$ ,  $\mathrm{pr}_{\mathcal{L}}^G$  and  $\mathrm{pr}_{\mathcal{L}}^{\mathcal{P}}$  denote the canonical projections.

**1.2.  $\mathrm{Ind}_{\mathcal{P}}^G \sigma$  and the principal series  $(C_{\sigma}^{\mathrm{an}}(\mathcal{P}, V), T_{\sigma})$ .** Let  $(V, \sigma)$  be a *locally analytic representation* (cf. [3] 3.1.5 and [11] §3) of  $H$  on a barreled locally convex Hausdorff  $K$ -vector space  $V$ , which means that the orbit maps are  $V$ -valued locally analytic functions; more precisely, for any  $v \in V$  there exists a BH-space  $W$  of  $V$  (that is, a Banach space  $W$  together with a continuous injection  $W \hookrightarrow V$ ) such that  $g \mapsto \sigma(g)v$  expands in a neighborhood of the unit element to a power series with  $W$ -coefficients (cf. [3]).

Observe that  $\sigma$  extends to a representation of  $\mathrm{P}$  via the projection

$$\mathrm{P} \rightarrow H, \quad \begin{pmatrix} {}^t D^{-1} & B \\ 0 & D \end{pmatrix} \mapsto D.$$

We consider the *parabolic induced representation*  $\mathrm{Ind}_{\mathcal{P}}^G \sigma$  whose underlying space is the space of  $V$ -valued locally analytic functions  $f$  on  $G$  satisfying

$$f(pg) = \sigma(p)f(g), \quad \text{for all } g \in G, p \in \mathrm{P};$$

$G$  acts by the right translation.

Because the homogeneous space  $\mathcal{L}$  is compact,  $\mathrm{Ind}_{\mathcal{P}}^G \sigma$  is a locally analytic representation of  $G$  ([3] 4.1.5).

Next, we give another description of  $\mathrm{Ind}_{\mathcal{P}}^G \sigma$ . Let  $C_{\sigma}^{\mathrm{an}}(\mathcal{P}, V)$  be the space of  $V$ -valued locally analytic functions  $\varphi$  on  $\mathcal{P}$  satisfying

$$\varphi(hX, hY) = \sigma(h)\varphi(X, Y), \quad \text{for all } (X, Y) \in \mathcal{P} \text{ and } h \in H.$$

We define the *principal series representation*  $(C_{\sigma}^{\mathrm{an}}(\mathcal{P}, V), T_{\sigma})$  of  $G$ :

$$(1.4) \quad (T_{\sigma}(g)\varphi)(X, Y) := \varphi((X, Y)g).$$

LEMMA 1.1.

- (1) The representation  $\text{Ind}_{\mathbb{P}}^G \sigma$  is (naturally) isomorphic to  $(C_{\sigma}^{\text{an}}(\mathcal{P}, V), T_{\sigma})$ .
- (2)  $\text{Ind}_{\mathbb{P}}^G \sigma$  is isomorphic to  $C^{\text{an}}(\mathcal{L}, V)$ .

PROOF. (1) From a locally analytic section  $\bar{\iota}$  of  $\text{pr}_{\mathcal{P}}^G$ , one obtains an isomorphism  $\bar{\iota}^{\circ} : \text{Ind}_{\mathbb{P}}^G \mathbf{1} \simeq C^{\text{an}}(\mathcal{P}, V)$ ,  $f \mapsto f \circ \bar{\iota}$  ([3] 4.3.1). By restriction,  $\bar{\iota}^{\circ}$  induces an isomorphism between  $\text{Ind}_{\mathbb{P}}^G \sigma$  and  $C_{\sigma}^{\text{an}}(\mathcal{P}, V)$ , which is independent on the choice of  $\bar{\iota}$ .  $G$ -equivariance is evident.

(2) A locally analytic section  $\bar{\iota}$  of  $\text{pr}_{\mathcal{L}}^G$  induces an isomorphism  $\bar{\iota}^{\circ} : \text{Ind}_{\mathbb{P}}^G \sigma \simeq C^{\text{an}}(\mathcal{L}, V)$  (ibid.). Q.E.D.

Because  $\mathcal{L}$  is compact,  $C^{\text{an}}(\mathcal{L}, V)$  is of compact type ([11] Lemma 2.1). By [11] Proposition 1.2, Theorem 1.3 and [9] Proposition 16.10, we have the following corollary.

COROLLARY 1.2. *Suppose that  $B$  is a closed subspace of  $C_{\sigma}^{\text{an}}(\mathcal{P}, V)$ . Then both  $B$  and  $C_{\sigma}^{\text{an}}(\mathcal{P}, V)/B$  are of compact type, in particular, they are reflexive, bornological, and complete;  $B_b^*$  and  $(C_{\sigma}^{\text{an}}(\mathcal{P}, V)/B)_b^*$  are nuclear Fréchet spaces.*

For technical needs, we fix a finite disjoint open covering  $\{\bar{\mathcal{U}}_k\}_k$  of  $\mathcal{L}$  satisfying:

- 1.  $\text{Sym}(n, \mathfrak{o}) \in \{\bar{\mathcal{U}}_k\}_k$ ;
- 2. each  $\bar{\mathcal{U}}_k$  is translated into  $\text{Sym}(n, \mathfrak{o})$  by an element  $g_k$  in  $G$ ;
- 3. Let  $\mathcal{U}_k := (\text{pr}_{\mathcal{L}}^{\mathcal{P}})^{-1}(\bar{\mathcal{U}}_k)$ . We define the analytic local section  $\iota_k : \bar{\mathcal{U}}_k \rightarrow \mathcal{U}_k$  of  $\text{pr}_{\mathcal{L}}^{\mathcal{P}}$  to be the  $g_k^{-1}$ -translation of the section

$$\iota_0 : \text{Sym}(n, F) \rightarrow \mathcal{U}_0, \quad z \mapsto (1, -z).$$

All the  $\iota_k$  give rise to a locally analytic section  $\iota$  of  $\text{pr}_{\mathcal{L}}^{\mathcal{P}}$ . Define  $\mathcal{K} := \iota(\mathcal{L})$ .

If the locally analytic sections  $\bar{\iota}$  and  $\bar{\tilde{\iota}}$  in the proof of Lemma 1.1 are compatible with  $\iota$ , in the sense that  $\bar{\tilde{\iota}} = \bar{\iota} \circ \iota$ , then Lemma 1.1 implies that  $\iota$  induces an isomorphism

$$(1.5) \quad \begin{aligned} \iota^{\circ} : C_{\sigma}^{\text{an}}(\mathcal{P}, V) &\rightarrow C^{\text{an}}(\mathcal{L}, V) \\ \varphi &\mapsto \varphi \circ \iota. \end{aligned}$$

**1.3. The  $p$ -adic Siegel upper half-space.** In this section, we first define a  $p$ -adic analogue of the Siegel upper half-space, which also generalizes the  $p$ -adic upper half-plane (cf. [2]), and then discuss some of their basic properties.

Let  $\mathbf{S}$  be the  $F$ -rigid analytic variety  $\mathbf{Sym}(n)$  that is isomorphic to the affine space  $\mathbb{A}_{/F}^{n(n+1)/2}$ . The underlying space of  $\mathbf{S}$  is  $\text{Sym}(n, F^{\text{alg}})$  (strictly speaking,  $\text{Sym}(n, F^{\text{alg}})/\text{Gal}(F^{\text{alg}}/F)$  (cf. [1]), but it is more convenient not to consider the Galois action in our situation).

DEFINITION 1.3. *Let*

$$\Sigma := \{Z \in \mathbf{S} : \det(XZ + Y) \neq 0 \text{ for any pair } (X, Y) \in \mathcal{P}\}.$$

$\Sigma$  is called the  $p$ -adic Siegel upper half-space.

Firstly, we show that  $\Sigma$  is nonempty.

LEMMA 1.4. *If  $Z$  is a diagonal matrix in  $\mathbf{S}$  whose diagonal entry  $Z_{ii}$  is of absolute value  $|\varpi|^{1/(n+1)^{k_i}}$ , with distinct positive integers  $k_i$ , then  $Z \in \Sigma$ .*

PROOF. One needs to show that  $\det(XZ + Y)$  is non-vanishing for any pair  $(X, Y) \in \mathcal{P}$ . Suitably multiplying a matrix in  $\mathbf{H}$  on the left and a permutation matrix on the right of  $X$  and  $Y$ , and conjugating  $Z$  by the same permutation matrix, we may assume that  $X = \begin{pmatrix} I_r & \widetilde{X} \\ 0 & 0 \end{pmatrix}$ , with  $r$  being the rank of  $X$ . Moreover, we write  $Y$  and  $Z$  in block matrices  $\begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix}$  and  $\begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix}$ , respectively. It follows from  $X {}^t Y = Y {}^t X$  that  $Y_3 + Y_4 {}^t \widetilde{X} = 0$  and  $Y_1 + Y_2 {}^t \widetilde{X}$  is symmetric. Then

$$\begin{aligned} XZ + Y &= \begin{pmatrix} Z_1 + Y_1 & \widetilde{X}Z_2 + Y_2 \\ -Y_4 {}^t \widetilde{X} & Y_4 \end{pmatrix} \\ &= \begin{pmatrix} I_r & 0 \\ 0 & Y_4 \end{pmatrix} \begin{pmatrix} Z_1 + Y_1 + \widetilde{X}Z_2 {}^t \widetilde{X} + Y_2 {}^t \widetilde{X} & \widetilde{X}Z_2 + Y_2 \\ 0 & I_{n-r} \end{pmatrix} \begin{pmatrix} I_r & 0 \\ -{}^t \widetilde{X} & I_{n-r} \end{pmatrix}. \end{aligned}$$

Therefore

$$\det(XZ + Y) = \det(Z_1 + \widetilde{X}Z_2 {}^t \widetilde{X} + Y_1 + Y_2 {}^t \widetilde{X}) \det Y_4.$$

In view of  $\text{rank}(X \ Y) = n$  and  $Y_3 = -Y_4 {}^t \widetilde{X}$ ,  $Y_4$  is invertible, namely  $\det Y_4 \neq 0$ . Clearly, the first determinant on the right is a nonzero polynomial in  $Z_{ii}$  with coefficients in  $F$ , and the degree of  $Z_{ii}$  in each term does not exceed  $n$ . By the assumptions on  $Z_{ii}$ , the terms that appear in the polynomial are of distinct absolute values, so the determinant is nonzero. In conclusion,  $\det(XZ + Y) \neq 0$ . Q.E.D.

In the following, we endow  $\Sigma$  with a structure of  $F$ -rigid analytic variety and show that  $\Sigma$  is an admissible open subset of  $\mathbf{S}$  and consequently an open rigid analytic subspace of  $\mathbf{S}$  (compare [10] §1 Proposition 1).

We define  $\mathcal{P}_0 = \text{pr}_{\mathcal{P}}^G(G_0)$ ;  $\mathcal{P}_0$  is compact. By Iwasawa's decomposition,  $G = P \cdot G_0$ ,  $\mathcal{P} = H \cdot \mathcal{P}_0$ , and therefore

$$\Sigma = \{Z \in \mathbf{S} : \det(XZ + Y) \neq 0 \text{ for any pair } (X, Y) \in \mathcal{P}_0\}.$$

For  $Z \in \mathbf{S}$ , let

$$|Z| := \max_{1 \leq i \leq j \leq n} \{1, |Z_{ij}|\}.$$

For a nonnegative integer  $m$  and a pair  $(X, Y) \in \mathcal{P}_0$ , we define

$$\mathbf{B}^-(m; X, Y) := \{Z \in \mathbf{S} : |\det(XZ + Y)| < |Z|^n |\varpi|^m\}.$$

LEMMA 1.5. *If  $m$  is a nonnegative integer and  $(X, Y), (X', Y') \in \mathcal{P}_0$  such that  $(X, Y) \equiv (hX', hY') \pmod{\varpi^{m+1}}$  for some  $h \in H_0$ , then*

$$\mathbf{B}^-(m; X, Y) = \mathbf{B}^-(m; X', Y').$$

PROOF. Obviously  $\mathbf{B}^-(m; X, Y) = \mathbf{B}^-(m; hX, hY)$ . We may therefore assume that  $(X, Y) \equiv (X', Y') \pmod{\varpi^{nm+1}}$ .

We choose  $\lambda \in (F^{\text{alg}})^\times$  such that  $|\lambda| = |Z|$ . With the observations that  $|\lambda^{-1}| \leq 1$  and  $|\lambda^{-1}Z_{ij}| \leq 1$ , one has

$$\begin{aligned} X \cdot \lambda^{-1}Z + Y \cdot \lambda^{-1} &\equiv X' \cdot \lambda^{-1}Z + Y' \cdot \lambda^{-1} \pmod{\varpi^{nm+1}}, \\ \det(XZ + Y) \cdot \lambda^{-n} &\equiv \det(X'Z + Y') \cdot \lambda^{-n} \pmod{\varpi^{nm+1}}, \end{aligned}$$

whence

$$|\det(XZ + Y)| |Z|^{-n} < |\varpi|^{nm} \Leftrightarrow |\det(X'Z + Y')| |Z|^{-n} < |\varpi|^{nm}.$$

Therefore  $\mathbf{B}^-(m; X, Y) = \mathbf{B}^-(m; X', Y')$ .

Q.E.D.

Define

$$\Sigma(m; X, Y) := \mathbf{S} - \mathbf{B}^-(m; X, Y) = \{Z \in \mathbf{S} : |\det(XZ + Y)| \geq |Z|^n |\varpi|^{nm}\}.$$

Let

$$\begin{aligned} \Sigma(m) &:= \bigcap_{(X,Y) \in \mathcal{P}_0} \Sigma(m; X, Y) \\ &= \left\{ Z \in \mathbf{S} : \left| \frac{\varpi^{nm}}{\det(XZ + Y)} \right| \leq 1, \left| \frac{\varpi^{nm} Z_{ij}^n}{\det(XZ + Y)} \right| \leq 1 \text{ for any } (X, Y) \in \mathcal{P}_0 \right\}. \end{aligned}$$

Let  $\mathcal{P}^{(m)}$  be any finite subset of  $\mathcal{P}_0$  containing  $(0, I_n)$  as well as a set of representatives in  $\mathcal{P}_0$  for  $H_0 \backslash \mathcal{P}_0 \pmod{\varpi^{nm+1}}$ . Then Lemma 1.5 implies that

$$\Sigma(m) = \bigcap_{(X,Y) \in \mathcal{P}^{(m)}} \Sigma(m; X, Y).$$

Denote  $\mathcal{P}_0^{(m)} = \mathcal{P}^{(m)} - \{(0, I_n)\}$ .

Observe that

$$\begin{aligned} \Sigma(m; 0, I_n) &= \{Z \in \mathbf{S} : |Z_{ij}| \leq |\varpi|^{-m}, 1 \leq i \leq j \leq n\} \\ &= \text{Sp}\left(F \left\langle \varpi^m Z_{ij} : 1 \leq i \leq j \leq n \right\rangle\right), \end{aligned}$$

is an admissible open affinoid subset of  $\mathbf{S}$ . Thus,  $\Sigma(m)$  is the intersection of a finite number of rational sub-domains of  $\Sigma(m; 0, I_n)$ :

$$\left\{ Z \in \Sigma(m; 0, I_n) : \left| \frac{\varpi^{nm}}{\det(XZ + Y)} \right| \leq 1, \left| \frac{\varpi^{nm} Z_{ij}^n}{\det(XZ + Y)} \right| \leq 1 \right\},$$

for all  $(X, Y) \in \mathcal{P}_0^{(m)}$ . Therefore  $\Sigma(m)$  is the affinoid variety:

$$(1.6) \quad \text{Sp}\left(F \left\langle \varpi^m Z_{ij} \right\rangle \left\langle \frac{\varpi^{nm}}{\det(XZ + Y)}, \frac{\varpi^{nm} Z_{ij}^n}{\det(XZ + Y)} : (X, Y) \in \mathcal{P}_0^{(m)} \right\rangle\right).$$

Now,  $\{\Sigma(m)\}_{m=0}^\infty$  forms an admissible affinoid covering of  $\Sigma$ . This gives rise to a rigid analytic variety structure on  $\Sigma$  (see [1] 9.3). According to [1] 9.1.2 Lemma 3 (compare [1] 9.1.4 Proposition 2), the following Proposition implies that  $\Sigma$  is an admissible open subset of  $\mathbf{S}$ .

PROPOSITION 1.6. *Any morphism from an affinoid variety to  $\mathbf{S}$  with image in  $\Sigma$  factors through some  $\Sigma(m)$ .*

PROOF. The argument is similar to the third proof of [10] §1 Proposition 1.

Let  $\mathbf{X}$  be an affinoid variety and  $\phi : \mathbf{X} \rightarrow \mathbf{S}$  be a morphism from  $\mathbf{X}$  to  $\mathbf{S}$  with image in  $\Sigma$ . For any  $(X, Y) \in \mathcal{P}_0$ ,

$$x \mapsto \frac{1}{\det(X\phi(x) + Y)}, \quad x \mapsto \frac{(\phi(x))_{ij}^n}{\det(X\phi(x) + Y)}$$

are  $F$ -rigid analytic functions on  $\mathbf{X}$ . By the maximum modulus principle ([1] §6.2 Proposition 4 (i)), there exists a positive integer  $m_{(X,Y)}$  such that

$$\max_{1 \leq i \leq j \leq n} \max_{x \in \mathbf{X}} \left\{ \left| \frac{1}{\det(X\phi(x) + Y)} \right|, \left| \frac{(\phi(x))_{ij}^n}{\det(X\phi(x) + Y)} \right| \right\} \leq |\varpi|^{-nm_{(X,Y)}}.$$

In other words,  $\phi(\mathbf{X}) \subset \Sigma(m_{(X,Y)}; X, Y)$ . In view of Lemma 1.5, one can choose  $m_{(X,Y)}$  to be locally constant. Therefore, there exists a positive integer  $m$  such that  $\phi(\mathbf{X}) \subset \Sigma(m)$  due to the compactness of  $\mathcal{P}_0$ . Q.E.D.

Let  $\mathcal{O}(\Sigma(m))$  denote the space of  $F$ -rigid analytic functions on  $\Sigma(m)$ ; it is an  $F$ -affinoid algebra with the supremum norm. From (1.6) one sees that  $\psi \in \mathcal{O}(\Sigma(m))$  admits an expansion in the form:

$$(1.7) \quad \psi(Z) = \sum_{(k_{(X,Y)}) \in \mathbb{N}^{\mathcal{P}(m)}} P_{(k_{(X,Y)})}(Z) \prod_{(X,Y) \in \mathcal{P}(m)} \det(XZ + Y)^{-k_{(X,Y)}},$$

where  $\mathbb{N}$  denotes the set of nonnegative integers,  $P_{(k_{(X,Y)})}(Z)$  are polynomials in  $Z_{ij}$  with  $F$ -coefficients, and the expansion converges with respect to the supremum norm  $\|\cdot\|_{\mathcal{O}(\Sigma(m))}$ . In particular,  $\det(XZ + Y)^{-1} \in \mathcal{O}(\Sigma(m))$  for any  $(X, Y) \in \mathcal{P}$ . Let  $\mathcal{O}(\Sigma)$  be the  $F$ -algebra of  $F$ -rigid analytic functions on  $\Sigma$ , which is the projective limit of  $\mathcal{O}(\Sigma(m))$ ,

$$\mathcal{O}(\Sigma) := \varprojlim_m \mathcal{O}(\Sigma(m)).$$

We endow  $\mathcal{O}(\Sigma)$  with the projective limit topology.

Let  $\mathcal{O}_K(\Sigma(m))$  and  $\mathcal{O}_K(\Sigma)$  denote  $\mathcal{O}(\Sigma(m)) \otimes_F K$  and  $\mathcal{O}(\Sigma) \otimes_F K$ , respectively. If one let  $\Sigma_K(m)$  and  $\Sigma_K$  denote the extension of the ground field  $K/F$  of  $\Sigma(m)$  and  $\Sigma$ , respectively ([1] §9.3.6), then  $\mathcal{O}_K(\Sigma(m))$  and  $\mathcal{O}_K(\Sigma)$  are the  $K$ -rigid analytic functions on  $\Sigma_K(m)$  and  $\Sigma_K$ , respectively.

PROPOSITION 1.7.  *$\mathcal{O}_K(\Sigma)$  is a nuclear  $K$ -Fréchet space.*

PROOF. According to [9] Corollary 16.6 and Proposition 19.9, it suffices to prove that  $\mathcal{O}_K(\Sigma(m))$  form a compact projective system.

Observe that  $\mathcal{O}_K(\Sigma(m))$  is generated by

$$(1.8) \quad \varpi^m Z_{ij}, \frac{\varpi^{nm}}{\det(XZ + Y)}, \frac{\varpi^{nm} Z_{ij}^n}{\det(XZ + Y)}, \quad 1 \leq i \leq j \leq n, (X, Y) \in \mathcal{P}_0^{(m)}.$$



Since

$$\sup_{Z \in \Sigma(m-1)} \sup_{\substack{(X,Y) \in \mathcal{P}_0^{(m)} \\ 1 \leq i \leq j \leq n}} \left\{ |\varpi^m Z_{ij}|, \left| \frac{\varpi^{nm}}{\det(XZ + Y)} \right|, \left| \frac{\varpi^{nm} Z_{ij}^n}{\det(XZ + Y)} \right| \right\} \leq |\varpi|.$$

By [12] Lemma 1.5, the transition homomorphism from  $\mathcal{O}_K(\Sigma(m))$  to  $\mathcal{O}_K(\Sigma(m-1))$  is compact. Q.E.D.

Clearly a compact projective system passes to closed subspaces. Moreover, a  $K$ -Fréchet space is the strong dual of a space of compact type if and only if it is nuclear ([11] Theorem 1.3).

**COROLLARY 1.8.** *Let  $\mathcal{N}$  be a closed subspace of  $\mathcal{O}_K(\Sigma)$ , then  $\mathcal{N}$  is a nuclear Fréchet space;  $\mathcal{N}_b^*$  is of compact type.*

**REMARK 1.9.** *If  $K$  is spherically complete, then Theorem 1.3 and Proposition 1.2 in [11] imply that  $\mathcal{O}_K(\Sigma)/\mathcal{N}$  is also a nuclear Fréchet space.*

All the generators (1.8) of  $\mathcal{O}(\Sigma(m))$  are  $F$ -rigid analytic functions on  $\Sigma(m')$ , for any  $m' \geq m$ , and therefore on  $\Sigma$  as well. Then we obtain the following proposition.

**PROPOSITION 1.10.**

- (1)  $\Sigma$  is a Stein space (cf. [4]), that is, the image of  $\mathcal{O}(\Sigma(m+1))$  under the transition homomorphism in  $\mathcal{O}(\Sigma(m))$  is dense for any nonnegative integer  $m$ .
- (2) The image of  $\mathcal{O}(\Sigma)$  under the transition homomorphism in  $\mathcal{O}(\Sigma(m))$  is dense.

Finally, we define a rigid analytic  $G$ -action on  $\Sigma$ :

$$gZ := (AZ + B)(CZ + D)^{-1}, \quad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G, Z \in \Sigma.$$

(1.1) is required to show that this is indeed a  $G$ -action. Moreover, we define the automorphy factor

$$j(g, Z) := (CZ + D).$$

From a straightforward computation, one verifies the automorphy (cocycle) relation

$$(1.9) \quad j(g_1 g_2, Z) = j(g_1, g_2 Z) j(g_2, Z).$$

**LEMMA 1.11.** *Let  $m$  be a nonnegative integer. Then for any  $g \in G_0$ ,*

$$g\Sigma(m) \subset \Sigma(nm).$$

**PROOF.** Let  $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_0$ ,  $Z \in \Sigma(m)$  and  $(X, Y) \in \mathcal{P}_0$ . We have  $(XA + YC, XB + YD) \in \mathcal{P}_0$ , whence

$$\begin{aligned} \frac{|Z|^n}{|\det(CZ + D)|} &\leq |\varpi|^{-nm}, \\ \frac{|Z|^n}{|\det((XA + YC)Z + (XB + YD))|} &\leq |\varpi|^{-nm}. \end{aligned}$$

By Cramer's rule,

$$\begin{aligned} gZ &= (AZ + B)(CZ + D)^{-1} \\ &= (AZ + B) \cdot \text{adj}(CZ + D) \frac{1}{\det(CZ + D)}, \end{aligned}$$

where  $\text{adj}(CZ + D)$  denotes the adjugate matrix of  $CZ + D$ . It is clear that  $\det(CZ + D)$  and all the entries of  $(AZ + B) \cdot \text{adj}(CZ + D)$  are polynomials in  $Z_{ij}$  with coefficients in  $\mathfrak{o}$  and degree  $\leq n$ , so

$$\begin{aligned} |\det(CZ + D)| &\leq |Z|^n, \\ |((AZ + B) \cdot \text{adj}(CZ + D))_{ij}| &\leq |Z|^n. \end{aligned}$$

Combining these,

$$\begin{aligned} &\frac{|gZ|^n}{|\det(X(gZ) + Y)|} \\ &= \frac{|\det(CZ + D)|}{|\det((XA + YC)Z + (XB + YD))|} \max \left\{ 1, \frac{|((AZ + B) \cdot \text{adj}(CZ + D))_{ij}|^n}{|\det(CZ + D)|^n} \right\} \\ &\leq \frac{|Z|^n}{|\det((XA + YC)Z + (XB + YD))|} \max \left\{ 1, \frac{|Z|^{n^2-n}}{|\det(CZ + D)|^{n-1}} \right\} \\ &\leq |\varpi|^{-n^2m}. \end{aligned}$$

Therefore  $g\Sigma(m) \subset \Sigma(nm)$ .

Q.E.D.

**1.4. Holomorphic discrete series  $(\mathcal{O}_\sigma(\Sigma), \pi_\sigma)$ .** We abbreviate  $\Sigma_K(m)(K)$  and  $\Sigma_K(K)$  to  $\Sigma(m)$  and  $\Sigma$ , respectively. Conventionally,  $\mathcal{O}_K(\Sigma(m))$  is described as the space of  $K$ -valued functions on  $\Sigma(m)$  with expansions of the form (1.7) that converge in the supremum norm of the  $K$ -valued function space on  $\Sigma_K(m)$ , and  $\mathcal{O}_K(\Sigma)$  as the space of  $K$ -valued functions on  $\Sigma$  whose restrictions on  $\Sigma(m)$  are functions in  $\mathcal{O}_K(\Sigma(m))$ . Furthermore, we abbreviate  $\mathcal{O}_K(\Sigma(m))$  and  $\mathcal{O}_K(\Sigma)$  to  $\mathcal{O}(\Sigma(m))$  and  $\mathcal{O}(\Sigma)$ , respectively.

Let  $(V, \sigma)$  be a  $d$ -dimensional  $K$ -rational representation of  $H$ . Let

$$\sigma(h) = \det(h)^{-s} P(h), \quad s \in \mathbb{N}, P \in M(d, K[h_{ij}]).$$

Let  $\mathcal{O}_\sigma(\Sigma(m)) := \mathcal{O}(\Sigma(m)) \otimes_K V$  and  $\mathcal{O}_\sigma(\Sigma) := \mathcal{O}(\Sigma) \otimes_K V$ . We define the *holomorphic (rigid analytic) discrete series representation*  $(\mathcal{O}_\sigma(\Sigma), \pi_\sigma)$  of  $G$ :

$$(1.10) \quad (\pi_\sigma(g)\psi)(Z) := \sigma(j(g^{-1}, Z))^{-1} \psi(g^{-1}Z), \quad \psi \in \mathcal{O}_\sigma(\Sigma), g \in G.$$

Since Proposition 1.6 implies that  $g^{-1}$  translates  $\Sigma(m)$  into some  $\Sigma(m')$ , it is not difficult to see that  $\pi_\sigma(g)\psi \in \mathcal{O}_\sigma(\Sigma)$ . This follows from showing that its coordinates have expansions of the form (1.7) and are bounded under the supremum norms  $\|\cdot\|_{\mathcal{O}(\Sigma(m))}$ . By the automorphy relation (1.9), one verifies that  $\pi_\sigma$  is a  $G$ -representation.

**PROPOSITION 1.12.**  $(\mathcal{O}_\sigma(\Sigma), \pi_\sigma)$  is continuous.

PROOF. Since  $\mathcal{O}_\sigma(\Sigma)$  is the projective limit of  $\mathcal{O}_\sigma(\Sigma(m))$ , it suffices to prove, for each  $m$ , the continuity of

$$\begin{aligned} G_0 \times \mathcal{O}_\sigma(\Sigma) &\rightarrow \mathcal{O}_\sigma(\Sigma(m)) \\ (g, \psi) &\mapsto (\pi_\sigma(g)\psi)|_{\Sigma(m)}. \end{aligned}$$

Moreover, according to Lemma 1.11,  $G_0\Sigma(m) \subset \Sigma(nm)$ , whence the above map factors through  $G_0 \times \mathcal{O}_\sigma(\Sigma(nm))$ . Thus we only need to consider the continuity of the map:

$$(1.11) \quad \begin{aligned} G_0 \times \mathcal{O}_\sigma(\Sigma(nm)) &\rightarrow \mathcal{O}_\sigma(\Sigma(m)) \\ (g, \psi) &\mapsto (\pi_\sigma(g)\psi)|_{\Sigma(m)}. \end{aligned}$$

For  $g \in G_0$ , entries of  $\sigma(j(g^{-1}, Z))^{-1}$  are  $\mathfrak{o}$ -coefficient polynomials, with  $Z_{ij}, \det(j(g^{-1}, Z))^{-1}$  and the coefficients of  $P$  viewed as variables. For  $Z \in \Sigma(m)$ , we have  $|Z_{ij}| \leq |\varpi|^{-m}$  and  $|\det(j(g^{-1}, Z))^{-1}| \leq |\varpi|^{-nm}$ , so there is a constant  $c > 0$  such that

$$\max_{g \in G_0} \max_{Z \in \Sigma(m)} \|\sigma(j(g^{-1}, Z))^{-1}\|_{\text{End}(V)} \leq c.$$

Therefore

$$(1.12) \quad \begin{aligned} &\max_{g \in G_0} \|\pi_\sigma(g)\psi\|_{\mathcal{O}_\sigma(\Sigma(m))} \\ &= \max_{g \in G_0} \max_{Z \in \Sigma(m)} \|(\pi_\sigma(g)\psi)(Z)\|_V \\ &\leq \max_{g \in G_0} \max_{Z \in \Sigma(m)} \|\sigma(j(g^{-1}, Z))^{-1}\|_{\text{End}(V)} \cdot \max_{g \in G_0} \max_{Z \in \Sigma(m)} \|\psi(g^{-1}Z)\|_V \\ &\leq c \max_{Z \in \Sigma(nm)} \|\psi(Z)\|_V \\ &= c \|\psi\|_{\mathcal{O}_\sigma(\Sigma(nm))}. \end{aligned}$$

So the map (1.11) is continuous. Q.E.D.

Now let  $U_0(\mathfrak{o})$  denote the parameterized open neighborhood of the unit element,  $\text{Sym}(n, \mathfrak{o}) \times H_0 \times \text{Sym}(n, \mathfrak{o}) \subset U_0 \cap G_0$ .

PROPOSITION 1.13. *For any  $\psi \in \mathcal{O}_\sigma(\Sigma(nm))$ , the orbit map*

$$\begin{aligned} U_0(\mathfrak{o}) &\rightarrow \mathcal{O}_\sigma(\Sigma(m)) \\ g &\mapsto (\pi_\sigma(g)\psi)|_{\Sigma(m)} \end{aligned}$$

*is an  $\mathcal{O}_\sigma(\Sigma(m))$ -valued analytic function (that is, can be expanded as a convergent power series with variables the coordinate parameters of  $U_0(\mathfrak{o})$  and coefficients in the Banach space  $\mathcal{O}_\sigma(\Sigma(m))$ ).*

PROOF. We first prove the following lemma.

LEMMA 1.14. *Let  $\psi \in \mathcal{O}_\sigma(\Sigma(nm))$ ,  $z \in \text{Sym}(n, \mathfrak{o})$  and  $h \in H_0$ .*

(1)  $\pi_\sigma \begin{pmatrix} I_n & z \\ 0 & I_n \end{pmatrix} \psi(Z) = \psi(Z-z)$  expands into a convergent power series in  $z_{ij}$  ( $1 \leq i \leq j \leq n$ ) with coefficients in  $\mathcal{O}_\sigma(\Sigma(m))$ ;

(2)  $\pi_\sigma \begin{pmatrix} {}^t h^{-1} & 0 \\ 0 & h \end{pmatrix} \psi(Z) = \sigma(h) \psi({}^t h Z h)$  expands into a convergent power series in  $h_{ij} - \delta_{ij}$  ( $1 \leq i, j \leq n$ ) with coefficients in  $\mathcal{O}_\sigma(\Sigma(m))$ , where  $\delta_{ij}$  is the Kronecker delta.

PROOF. (1) We consider the ring  $\mathcal{O}(\Sigma(m))[[z]]$  of formal power series  $\varphi(z)$  in  $z_{ij}$  with coefficients in  $\mathcal{O}(\Sigma(m))$ ;  $\varphi(z)$  is expressed as

$$\varphi(z) = \sum_{\underline{r} \in \text{Sym}(n, \mathbb{N})} \alpha_{\underline{r}} \cdot \underline{z}^{\underline{r}}, \quad \alpha_{\underline{r}} \in \mathcal{O}(\Sigma(m)), \quad \underline{z}^{\underline{r}} := \prod_{1 \leq i \leq j \leq n} z_{ij}^{r_{ij}}.$$

If the constant term  $\alpha_{\underline{0}}$  is invertible in  $\mathcal{O}(\Sigma(m))$ , then  $\varphi(z) \in \mathcal{O}(\Sigma(m))[[z]]^\times$ . In particular, for  $(X, Y) \in \mathcal{P}$ , the constant term in the expansion of  $\det(X(Z-z) + Y)$ , which is  $\det(XZ + Y)$ , is invertible in  $\mathcal{O}(\Sigma(m))$ , whence  $\det(X(Z-z) + Y)^{-1}$  belongs to  $\mathcal{O}(\Sigma(m))[[z]]$ .

In view of the expansion form (1.7), the discussions in the last paragraph imply that each coordinate of  $\psi(Z-z)$  expands into a formal power series in  $z_{ij}$  whose coefficients are series in  $\mathcal{O}(\Sigma(m))$ . It follows from the estimates in (1.12) that

1. the coefficients are indeed convergent series in  $\mathcal{O}(\Sigma(m))$  so that each coordinate of  $\psi(Z-z)$  belongs to  $\mathcal{O}(\Sigma(m))[[z]]$ ,
2. the  $\mathcal{O}(\Sigma(m))$ -coefficient formal power series expansion of  $\psi(Z-z)$  converges in  $\mathcal{O}_\sigma(\Sigma(m))$  for all  $z \in \text{Sym}(n, \mathfrak{o})$ .

(2) can be proven similarly.

Q.E.D.

From (1.3), we see that  $g \in U_0(\mathfrak{o})$  decomposes in  $G_0$  into

$$\begin{pmatrix} I_n & z_1 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} {}^t h^{-1} & 0 \\ 0 & h \end{pmatrix} \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} I_n & z_2 \\ 0 & I_n \end{pmatrix}$$

where  $z_1, z_2 \in \text{Sym}(n, \mathfrak{o})$  and  $h \in H_0$ . Lemma 1.14 and (1.12) imply that  $\pi_\sigma(g)\psi$  expands into a convergent  $\mathcal{O}_\sigma(\Sigma(m))$ -coefficient power series whose variables are the coordinate parameters of  $U_0(\mathfrak{o})$ .

Q.E.D.

COROLLARY 1.15. *The power series expansion of  $\det(Z-z)^{-1}$  on  $\text{Sym}(n, \mathfrak{o})$  converges in  $\mathcal{O}(\Sigma(m))$ , or equivalently,  $\det(Z-z)^{-1}$  expands into a power series*

$$\sum_{\underline{r} \in \text{Sym}(n, \mathbb{N})} \alpha_{\underline{r}}(Z) \cdot \underline{z}^{\underline{r}}, \quad \alpha_{\underline{r}} \in \mathcal{O}(\Sigma(m)),$$

such that  $\lim_{|\underline{r}| \rightarrow \infty} \|\alpha_{\underline{r}}\|_{\mathcal{O}(\Sigma(m))} = 0$ , with the notation  $|\underline{r}| = \sum_{1 \leq i \leq j \leq n} r_{ij}$ .

Next, we consider the adjoint representation  $\pi_\sigma^*$  of  $G$  on  $\mathcal{O}_\sigma(\Sigma)_b^* \cong \varinjlim_m \mathcal{O}_\sigma(\Sigma(m))_b^*$ .

The transition homomorphisms  $\mathcal{O}_\sigma(\Sigma(m))_b^* \rightarrow \mathcal{O}_\sigma(\Sigma)_b^*$  are injective (Proposition 1.10 (2)). Lemma 1.11 implies that, for any  $g \in G_0$ ,  $\pi_\sigma^*(g)$  maps  $\mathcal{O}_\sigma(\Sigma(m))_b^*$  into  $\mathcal{O}_\sigma(\Sigma(nm))_b^*$  via

$$\langle \psi, \pi_\sigma^*(g)\mu \rangle = \langle (\pi_\sigma(g^{-1})\psi)|_{\Sigma(m)}, \mu \rangle, \quad \mu \in \mathcal{O}_\sigma(\Sigma(m))^*, \psi \in \mathcal{O}_\sigma(\Sigma(nm)).$$

It is easy to deduce from Proposition 1.13 that, for any  $\mu \in \mathcal{O}_\sigma(\Sigma(m))^*$ , the orbit map

$$\begin{aligned} U_0(\mathfrak{o})^{-1} &\rightarrow \mathcal{O}_\sigma(\Sigma(nm))_b^* \\ g &\mapsto \pi_\sigma^*(g)\mu \end{aligned}$$

is an  $\mathcal{O}_\sigma(\Sigma(nm))_b^*$ -valued analytic function. Therefore we have the following corollary.

**COROLLARY 1.16.**  *$(\mathcal{O}_\sigma(\Sigma)_b^*, \pi_\sigma^*)$  is locally analytic.*

Finally, we study the *de Rham complex*  $\Omega^r(\Sigma)$  of rigid analytic exterior differential forms. Explicitly, let  $0 \leq r \leq n(n+1)/2$ ,

$$\begin{aligned}\Omega_K^1 &:= \bigoplus_{1 \leq i \leq j \leq n} K dZ_{ij}, \\ \Omega_K^r &:= \bigwedge^r \Omega_K^1(\Sigma), \\ \Omega^r(\Sigma) &:= \mathcal{O}(\Sigma) \otimes_K \Omega_K^r.\end{aligned}$$

As interesting examples, we show that the spaces  $\Omega^r(\Sigma)$  as  $G$ -representations are holomorphic discrete series of  $G$  (compare [8] §3).

We define a  $K$ -rational representation  $\sigma_1$  of  $H$  on  $\Omega_K^1$ :

$$\sigma_1(h)dZ_{ij} := \sum_{1 \leq k < \ell \leq n} (h_{ik}h_{j\ell} + h_{i\ell}h_{jk})dZ_{k\ell} + \sum_{k=1}^n h_{ik}h_{jk}dZ_{kk},$$

or succinctly,

$$\sigma_1(h)dZ = h \cdot dZ \cdot {}^t h, \quad dZ := (dZ_{ij}).$$

Let  $\sigma_r := \bigwedge^r \sigma_1$ .

1. For  $g = \begin{pmatrix} I_n & z \\ 0 & I_n \end{pmatrix}$ , we have  $g \cdot dZ = d(Z - z) = dZ$ .
2. For  $g = \begin{pmatrix} {}^t h^{-1} & 0 \\ 0 & h \end{pmatrix}$ , we have  $g \cdot dZ = d(hZ {}^t h) = h \cdot dZ \cdot {}^t h = \sigma_1(h)dZ$ .
3. For  $g = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ , we have  $g \cdot dZ = d(-Z^{-1}) = Z^{-1} \cdot dZ \cdot Z^{-1} = \sigma_1(Z^{-1})dZ$  due to the identity  $d(Z^{-1}) \cdot Z + Z^{-1} \cdot dZ = 0$ .

In view of the decomposition (1.3) of  $U_0$ , the discussions above imply that the action of  $G$  on  $\Omega^r(\Sigma)$  coincides with  $\pi_{\sigma_r}$  on  $U_0$  and hence on  $G$  as  $U_0$  is dense in  $G$ .

**PROPOSITION 1.17.** *Let  $1 \leq r \leq n(n+1)/2$  and  $\sigma_r$  be defined above. The  $G$ -action on  $\Omega^r(\Sigma)$  coincides with  $\pi_{\sigma_r}$ .*

## 2. Duality

In the following, we assume that  $K$  is spherically complete. Let  $(V, \sigma)$  be a  $d$ -dimensional  $K$ -rational representation of  $H$ . We choose a basis  $v_1, \dots, v_d$  of  $V$ ; we denote by  $v_1^*, \dots, v_d^*$  the corresponding dual basis of the dual space  $V^*$ . Let  $(V^*, \sigma^*)$  denote the dual representation of  $(V, \sigma)$ .

**2.1. The duality operator  $I_\sigma$ .** For  $Z \in \Sigma$  and  $v^* \in V^*$ , let  $\varphi_{Z,v^*}$  be the  $V^*$ -valued locally analytic function on  $\mathcal{P}$ :

$$(2.1) \quad \varphi_{Z,v^*}(X, Y) := \sigma^*(XZ + Y)v^*.$$

Let  $B_{\sigma^*}^0(\mathcal{P}, V^*)$  be the subspace of  $C_{\sigma^*}^{\text{an}}(\mathcal{P}, V^*)$  spanned by  $\varphi_{Z,v^*}$ ,  $B_{\sigma^*}(\mathcal{P}, V^*)$  the closure of  $B_{\sigma^*}^0(\mathcal{P}, V^*)$ . Clearly  $B_{\sigma^*}(\mathcal{P}, V^*)$  is  $G$ -invariant.

For any continuous linear functional  $\xi \in B_{\sigma^*}(\mathcal{P}, V^*)^*$ , we define a  $V$ -valued function on  $\Sigma$ :

$$(2.2) \quad I_\sigma(\xi)(Z) := \sum_{k=1}^d \langle \varphi_{Z,v_k^*}, \xi \rangle v_k, \quad Z \in \Sigma.$$

One verifies that  $I_\sigma(\xi)$  is independent of the choice of the basis  $\{v_k\}_{k=1}^d$ . Evidently,  $I_\sigma$  is injective.

LEMMA 2.1.  $I_\sigma$  is  $G$ -equivariant, that is,

$$I_\sigma(T_{\sigma^*}^*(g)\xi) = \pi_\sigma(g)I_\sigma(\xi),$$

for any  $g \in G$ .

PROOF. Let  $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$ . We have

$$\begin{aligned} I_\sigma(T_{\sigma^*}^*(g)\xi)(Z) &= \sum_{k=1}^d \langle \varphi_{Z,v_k^*}, T_{\sigma^*}^*(g)\xi \rangle v_k \\ &= \sum_{k=1}^d \langle T_{\sigma^*}(g^{-1})\varphi_{Z,v_k^*}, \xi \rangle v_k \\ &= \sum_{k=1}^d \langle \sigma^*(X({}^t DZ - {}^t B) + Y(-{}^t CZ + {}^t A))v_k^*, \xi \rangle v_k \\ &= \sigma(j(g^{-1}, Z))^{-1} \left( \sum_{k=1}^d \langle \sigma^*(X \cdot g^{-1}Z + Y)(v_{k,g}^*), \xi \rangle v_{k,g} \right) \\ &= (\pi_\sigma(g)I_\sigma(\xi))(Z), \end{aligned}$$

where  $v_{k,g} = \sigma(j(g^{-1}, Z))v_k$ .

Q.E.D.

PROPOSITION 2.2.

(1) For any continuous linear functional  $\xi \in B_{\sigma^*}(\mathcal{P}, V^*)^*$ ,  $I_\sigma(\xi)$  is a  $V$ -valued rigid analytic function on  $\Sigma$ .

(2)  $I_\sigma$  is a continuous homomorphism of  $G$ -representations from  $(B_{\sigma^*}(\mathcal{P}, V^*)^*)_b, T_{\sigma^*}^*)$  to  $(\mathcal{O}_\sigma(\Sigma), \pi_\sigma)$ .

PROOF. Let  $i$  denote the inclusion:  $B_{\sigma^*}(\mathcal{P}, V^*) \hookrightarrow C_{\sigma^*}^{\text{an}}(\mathcal{P}, V^*)$ , and  $i^*$  be its adjoint operator. Because of our assumption that  $K$  is spherically complete, the Hahn-Banach Theorem ([9] Corollary 9.4) implies that  $i^*$  is surjective. Since  $C_{\sigma^*}^{\text{an}}(\mathcal{P}, V^*)^*_b$  and  $B_{\sigma^*}(\mathcal{P}, V^*)^*_b$  are

Fréchet spaces (Corollary 1.2),  $i^*$  is open (from the open mapping theorem ([9] Proposition 8.6)). Consequently, the continuity of  $I_\sigma \circ i^*$  implies that of  $I_\sigma$ . Therefore, (1) and (2) are equivalent to:

(1')  $I_\sigma \circ i^*(\xi) \in \mathcal{O}_\sigma(\Sigma)$  for any  $\xi \in C_{\sigma^*}^{\text{an}}(\mathcal{P}, V^*)^*$ ;

(2')  $I_\sigma \circ i^* : (C_{\sigma^*}^{\text{an}}(\mathcal{P}, V^*)_b^*, T_{\sigma^*}^*) \rightarrow (\mathcal{O}_\sigma(\Sigma), \pi_\sigma)$  is a continuous homomorphism of G-representations.

For brevity, we still denote  $I_\sigma \circ i^*$  by  $I_\sigma$ . For  $\xi \in C_{\sigma^*}^{\text{an}}(\mathcal{P}, V^*)^*$ , we write  $I_\sigma(\xi)$  in the form of integral:

$$\begin{aligned} I_\sigma(\xi)(Z) &= \sum_{k=1}^d \int_{\mathcal{P}} \varphi_{Z;v_k^*} d\xi \cdot v_k \\ &= \sum_{k=1}^d \sum_{\kappa} \int_{\mathcal{U}_\kappa} \varphi_{Z;v_k^*} d\xi \cdot v_k \\ &= \sum_{\kappa} \pi_\sigma(g_\kappa) \left( \sum_{k=1}^d \int_{\mathcal{U}_\kappa \cdot g_\kappa} \varphi_{Z;(v_{k;g_\kappa})^*} d(T_{\sigma^*}(g_\kappa^{-1})\xi) \cdot v_{k;g_\kappa} \right), \end{aligned}$$

where the disjoint open covering  $\{\mathcal{U}_\kappa\}_\kappa$  of  $\mathcal{P}$  and  $g_\kappa$  are defined in §1.2, and  $v_{k;g_\kappa}$  is defined in the proof of Lemma 2.1. Therefore it suffices to consider

$$(2.3) \quad \sum_{k=1}^d \int_{\mathcal{U}} \varphi_{Z;v_k^*} d\xi' \cdot v_k.$$

where  $\mathcal{U}$  are taken to be  $\mathcal{U}_\kappa \cdot g_\kappa$  and  $\xi'$  is the image of  $\xi$  under  $C_{\sigma^*}^{\text{an}}(\mathcal{P}, V^*)_b^* \rightarrow C_{\sigma^*}^{\text{an}}(\mathcal{U}, V^*)_b^*$ .

For the open subset  $\overline{\mathcal{U}} = \text{pr}_{\mathcal{L}}^{\mathcal{P}}(\mathcal{U})$  of  $\text{Sym}(n, \mathfrak{o})$ , we have the isomorphism induced from the section  $\iota_0$  (compare (1.5)):

$$(2.4) \quad C_{\sigma^*}^{\text{an}}(\mathcal{U}, V^*)_b^* \cong C^{\text{an}}(\overline{\mathcal{U}}, V^*)_b^*.$$

Then (2.3) is equal to

$$\overline{I}_{\sigma, \overline{\mathcal{U}}}(\overline{\xi})(Z) := \sum_{k=1}^d \int_{\overline{\mathcal{U}}} (\sigma^*(Z - z)v_k^*) d\overline{\xi}(z) \cdot v_k,$$

where  $\overline{\xi}$  is the image of  $\xi'$  in  $C^{\text{an}}(\overline{\mathcal{U}}, V^*)_b^*$  via the isomorphism (2.4). Therefore, it suffices to prove that  $\overline{I}_{\sigma, \overline{\mathcal{U}}}(\overline{\xi})$  is rigid analytic on  $\Sigma(m)$ , and that the map

$$\begin{aligned} C^{\text{an}}(\overline{\mathcal{U}}, V^*)_b^* &\rightarrow \mathcal{O}_\sigma(\Sigma(m)) \\ \overline{\xi} &\mapsto \overline{I}_{\sigma, \overline{\mathcal{U}}}(\overline{\xi})|_{\Sigma(m)} \end{aligned}$$

is continuous (G-equivariance is already proven in Lemma 2.1).

As  $\sigma^*$  is algebraic, there is a nonnegative integer  $t$  and polynomials  $Q_{k\ell}$  ( $1 \leq k, \ell \leq d$ ) in  $h_{ij}$  ( $1 \leq i, j \leq n$ ) with coefficients in  $K$ , such that

$$\sigma^*(h)v_k^* = \sum_{\ell=1}^d \det(h)^{-t} Q_{k\ell}(h)v_\ell^*.$$

We expand

$$\det(Z - z)^{-t} Q_{k\ell}(Z - z) = \sum_{\underline{L}} \alpha_{\underline{L}, k\ell}(Z) \cdot \underline{z}^{\underline{L}}.$$

It is evident from Corollary 1.15 that  $\alpha_{\underline{L}, k\ell} \in \mathcal{O}(\Sigma(m))$  and

$$(2.5) \quad \lim_{|\underline{L}| \rightarrow \infty} \|\alpha_{\underline{L}, k\ell}\|_{\mathcal{O}(\Sigma(m))} = 0.$$

Moreover, there is a constant  $c_m > 0$ , depending only on  $m, \sigma$  and  $\{v_k\}_{k=1}^d$ , such that

$$(2.6) \quad \|\alpha_{\underline{L}, k\ell}\|_{\mathcal{O}(\Sigma(m))} \leq c_m.$$

Then

$$(2.7) \quad \begin{aligned} \bar{I}_{\sigma, \bar{\mathcal{U}}}(\bar{\xi})(Z) &= \sum_{k, \ell=1}^d \int_{\bar{\mathcal{U}}} \det(Z - z)^{-t} Q_{k\ell}(Z - z) d\bar{\xi}(z) \cdot v_k \\ &= \sum_{k=1}^d \left( \sum_{\ell=1}^d \sum_{\underline{L}} \left( \int_{\bar{\mathcal{U}}} \underline{z}^{\underline{L}} \cdot v_{\ell}^* d\bar{\xi}(z) \right) \cdot \alpha_{\underline{L}, k\ell}(Z) \right) v_k. \end{aligned}$$

Since  $\|\underline{z}^{\underline{L}}\|_{C^{\text{an}}(\bar{\mathcal{U}})} \leq 1$ , we have

$$(2.8) \quad \left| \int_{\bar{\mathcal{U}}} \underline{z}^{\underline{L}} \cdot v_{\ell}^* d\bar{\xi}(z) \right| \leq \|v_{\ell}^*\|_{V^*} \cdot \|\bar{\xi}\|_{C^{\text{an}}(\bar{\mathcal{U}}, V^*)_b}.$$

In conclusion, (2.5) and (2.8) imply that the expansion (2.7) of  $\bar{I}_{\sigma, \bar{\mathcal{U}}}(\bar{\xi})$  converges in  $\mathcal{O}_{\sigma}(\Sigma(m))$ , whereas (2.6) and (2.8) imply

$$\|\bar{I}_{\sigma, \bar{\mathcal{U}}}(\bar{\xi})\|_{\mathcal{O}_{\sigma}(\Sigma(m))} \leq \max_{1 \leq k, \ell \leq d} c_m \|v_{\ell}^*\|_{V^*} \|v_k\|_V \cdot \|\bar{\xi}\|_{C^{\text{an}}(\bar{\mathcal{U}}, V^*)_b}.$$

The continuity follows.

Q.E.D.

**2.2. The duality operator  $J_{\sigma}$  and the image of  $I_{\sigma}$ .** Let  $\mathcal{N}_{\sigma}(\Sigma)$  denote the image of  $I_{\sigma}$ . In this section, we propose to determine  $\mathcal{N}_{\sigma}(\Sigma)$ . For this, we introduce  $J_{\sigma}$ , the adjoint operator of  $I_{\sigma}$ : an injective continuous linear operator from  $\mathcal{N}_{\sigma}(\Sigma)_b^*$  to  $(B_{\sigma^*}(\mathcal{P}, V^*)_b^*)_b^* \cong B_{\sigma^*}(\mathcal{P}, V^*)$  ( $B_{\sigma^*}(\mathcal{P}, V^*)$  is reflexive according to Corollary 1.2).

First, we need to find the formula for  $J_{\sigma}$ .

For any  $\mu \in \mathcal{N}_{\sigma}(\Sigma)^*$  and  $\xi \in B_{\sigma^*}(\mathcal{P}, V^*)^*$ , we have

$$(2.9) \quad \langle J_{\sigma}(\mu), \xi \rangle = \langle I_{\sigma}(\xi), \mu \rangle.$$

For  $(X, Y) \in \mathcal{P}$  and  $v \in V$ , we define the Dirac distribution  $\xi_{(X, Y), v}$ , which is a continuous linear functional of  $B_{\sigma^*}(\mathcal{P}, V^*)$ , as follows:

$$\langle \varphi, \xi_{(X, Y), v} \rangle = \langle v, \varphi(X, Y) \rangle_V, \quad \varphi \in B_{\sigma^*}(\mathcal{P}, V^*),$$

and a  $V$ -valued rigid analytic function  $\psi_{(X, Y), v}$  on  $\Sigma$ :

$$(2.10) \quad \psi_{(X, Y), v}(Z) := \sigma(XZ + Y)^{-1}v.$$

LEMMA 2.3.

$$I_{\sigma}(\xi_{(X, Y), v}) = \psi_{(X, Y), v}.$$



PROOF. By definition (2.2),

$$\begin{aligned}
 (I_\sigma(\xi_{(X,Y,v)}))(Z) &= \sum_{k=1}^r \langle \varphi_{Z,v_k^*}, \xi_{(X,Y,v)} \rangle v_k \\
 &= \sum_{k=1}^r \langle v, \sigma^*(XZ + Y)v_k^* \rangle_V \cdot v_k \\
 &= \sum_{k=1}^r \langle \sigma(XZ + Y)^{-1}v, v_k^* \rangle_V \cdot v_k \\
 &= \sigma(XZ + Y)^{-1}v = \psi_{(X,Y,v)}(Z).
 \end{aligned}$$

Q.E.D.

Let  $\mathcal{N}_\sigma^0(\Sigma)$  denote the subspace of  $\mathcal{O}_\sigma(\Sigma)$  spanned by  $\psi_{(X,Y,v)}$  for all  $(X, Y) \in \mathcal{P}$  and  $v \in V$ . Clearly  $\mathcal{N}_\sigma^0(\Sigma)$  is  $G$ -invariant. Lemma 2.3 implies  $\mathcal{N}_\sigma^0(\Sigma) \subset \mathcal{N}_\sigma(\Sigma)$ .

PROPOSITION 2.4. *For any continuous linear functional  $\mu \in \mathcal{N}_\sigma(\Sigma)^*$ , we have*

$$(2.11) \quad J_\sigma(\mu)(X, Y) = \sum_{k=1}^d \langle \psi_{(X,Y,v_k)}, \mu \rangle v_k^*.$$

PROOF. We have

$$\begin{aligned}
 \sum_{k=1}^r \langle \psi_{(X,Y,v_k)}, \mu \rangle v_k^* &= \sum_{k=1}^r \langle I_\sigma(\xi_{(X,Y,v_k)}), \mu \rangle v_k^* && \text{(Lemma 2.3)} \\
 &= \sum_{k=1}^r \langle J_\sigma(\mu), \xi_{(X,Y,v_k)} \rangle v_k^* && \text{(Duality formula (2.9))} \\
 &= \sum_{k=1}^r \langle v_k, J_\sigma(\mu)(X, Y) \rangle_V \cdot v_k^* = J_\sigma(\mu)(X, Y).
 \end{aligned}$$

Q.E.D.

It follows from (2.11) that  $J_\sigma$  factors through  $\mathcal{N}_\sigma^0(\Sigma)^*$  and (2.11) defines an injection from  $\mathcal{N}_\sigma^0(\Sigma)_b^*$  to  $B_{\sigma^*}(\mathcal{P}, V^*)$ . Because  $J_\sigma$  is injective and  $\mathcal{N}_\sigma(\Sigma)_b^* \rightarrow \mathcal{N}_\sigma^0(\Sigma)_b^*$  is surjective (the Hahn-Banach Theorem), we have  $\mathcal{N}_\sigma^0(\Sigma)_b^* = \mathcal{N}_\sigma(\Sigma)_b^*$ . The following lemma then follows from the Hahn-Banach theorem.

LEMMA 2.5.  $\mathcal{N}_\sigma^0(\Sigma)$  is dense in  $\mathcal{N}_\sigma(\Sigma)$ .

THEOREM 2.6.

- (1)  $I_\sigma$  is an isomorphism from  $B_{\sigma^*}(\mathcal{P}, V^*)_b^*$  onto  $\mathcal{N}_\sigma(\Sigma)$ .
- (2)  $\mathcal{N}_\sigma(\Sigma)$  is the closure of  $\mathcal{N}_\sigma^0(\Sigma)$  in  $\mathcal{O}_\sigma(\Sigma)$ .

PROOF. Let  $B(\mathcal{L}, V^*) := \iota^\circ(B_{\sigma^*}(\mathcal{P}, V^*))$ . We still denote  $\iota^\circ|_{B_{\sigma^*}(\mathcal{P}, V^*)}$  by  $\iota^\circ$ .

Let  $\mathcal{I}$  be any (finite) disjoint open chart covering  $\{\overline{\mathcal{U}}_i\}_i$  of  $\mathcal{L}$ . We recall that  $C^{\text{an}}(\mathcal{L}, V^*)$  is defined as the inductive limit of the  $K$ -Banach algebra  $E_{\mathcal{I}}(\mathcal{L}, V^*) = \prod_i \mathcal{O}(\overline{\mathcal{U}}_i, V^*)$ , indexed with all the  $\mathcal{I}$ , where  $\mathcal{O}(\overline{\mathcal{U}}_i, V^*)$  denotes the space of  $K$ -analytic functions on  $\overline{\mathcal{U}}_i$  (cf. [3] 2.1.10 and [11] §2). The inductive limit structure is naturally induced onto  $B(\mathcal{L}, V^*)$ ,

that is,  $B(\mathcal{L}, V^*) = \varinjlim_I E_I(\mathcal{L}, V^*)$ . Moreover, the dual space  $B(\mathcal{L}, V^*)_b^*$  is the projective limit of  $E_I(\mathcal{L}, V^*)_b^*$ .

Let  $\mathcal{N}_\sigma^0(\Sigma(m))$  be the image of  $\mathcal{N}_\sigma^0(\Sigma)$  in  $\mathcal{O}_\sigma(\Sigma(m))$ .

Considering  $\pi_\sigma(g^{-1})v_k$ , we see that the map  $(X, Y) \mapsto \psi_{(X,Y),v_k}$  is an  $\mathcal{O}_\sigma(\Sigma(m))$ -valued locally analytic map on  $\mathcal{P}$  (see Proposition 1.13). Define

$$r_m = \min_{1 \leq k \leq d} \inf_{(X,Y) \in \mathcal{K}} \|\psi_{(X,Y),v_k}\|_{\mathcal{O}_\sigma(\Sigma(m))}$$

Since  $\mathcal{K}$  is compact,  $r_m$  is positive. Let  $\mathcal{L}$  be the lattice  $\sum_{k=1}^d \sum_{(X,Y) \in \mathcal{K}} \mathfrak{o}_K \cdot \psi_{(X,Y),v_k}$  in  $\mathcal{N}_\sigma^0(\Sigma)$ .

For each  $m$ , the image of  $\mathcal{L}$  in  $\mathcal{N}_\sigma^0(\Sigma(m))$  contains the ball of radius  $r_m$  centered at zero, and therefore the interior of  $\mathcal{L}$  is a nontrivial open lattice.

Consider

$$\begin{aligned} (\iota^{\circ-1})^* \circ I_\sigma^{-1}|_{\mathcal{N}_\sigma^0(\Sigma)} : \mathcal{N}_\sigma^0(\Sigma) &\rightarrow B(\mathcal{L}, V^*)_b^* \\ \psi_{(X,Y),v} &\mapsto (\iota^{\circ-1})^*(\xi_{(X,Y),v}), \end{aligned}$$

For  $(X, Y) \in \mathcal{K}$ ,

$$\begin{aligned} \|(\iota^{\circ-1})^*(\xi_{(X,Y),v})\|_{E_I(\mathcal{L}, V^*)_b^*} &= \max_{\bar{\varphi} \in E_I(\mathcal{L}, V^*)} \frac{\langle \bar{\varphi}, (\iota^{\circ-1})^*(\xi_{(X,Y),v}) \rangle}{\|\bar{\varphi}\|_{E_I(\mathcal{L}, V^*)}} \\ &= \max_{\varphi \in \iota^{\circ-1}(E_I(\mathcal{L}, V^*))} \frac{\langle \varphi, \xi_{(X,Y),v} \rangle}{\|\iota^\circ(\varphi)\|_{E_I(\mathcal{L}, V^*)}} \\ &= \max_{\varphi \in \iota^{\circ-1}(E_I(\mathcal{L}, V^*))} \frac{\langle v, \varphi(X, Y) \rangle_V}{\max_{(X', Y') \in \mathcal{K}} \|\varphi(X', Y')\|_{V^*}} \\ &\leq \|v\|_V. \end{aligned}$$

Therefore the image of  $\mathcal{L}$  under  $(\iota^{\circ-1})^* \circ I_\sigma^{-1}|_{\mathcal{N}_\sigma^0(\Sigma)}$  in  $B(\mathcal{L}, V^*)_b^*$  is bounded, since its image in  $E_I(\mathcal{L}, V^*)_b^*$  are all norm-bounded by  $\max_{1 \leq k \leq d} \|v_k\|_V$ . Because  $\mathcal{N}_\sigma^0(\Sigma)$  is metrizable, it is bornological ([9] Proposition 6.14), and therefore  $I_\sigma^{-1}|_{\mathcal{N}_\sigma^0(\Sigma)}$  is continuous ([9] Proposition 6.13).  $\mathcal{N}_\sigma^0(\Sigma)$  is isomorphic to  $I_\sigma^{-1}(\mathcal{N}_\sigma^0(\Sigma))$ , then their completions are isomorphic, which, in view of Lemma 2.5, must be  $\mathcal{N}_\sigma(\Sigma)$  and  $B_{\sigma^*}(\mathcal{P}, V^*)_b^*$ , respectively. Q.E.D.

**COROLLARY 2.7.**  *$J_\sigma$  is an isomorphism of G-representations from  $(\mathcal{N}_\sigma(\Sigma)_b^*, \pi_\sigma^*)$  onto  $(B_{\sigma^*}(\mathcal{P}, V^*), T_{\sigma^*})$ .*

**REMARK 2.8.** *We conjecture that  $(\mathcal{N}_\sigma(\Sigma), \pi_\sigma)$  and  $(B_{\sigma^*}(\mathcal{P}, V^*), T_{\sigma^*})$  are topologically irreducible G-representations if  $\sigma$  is irreducible. These are conjectured and claimed by Morita for  $\mathrm{SL}(2, F)$  ([6] Corollary after Theorem 3 and [7] Theorem 1 (i).) However, there is a serious gap in his proof of [7] Proposition 3. Schneider and Teitelbaum gave the first valid proof of [7] Theorem 1 (i) in [11] when  $F = \mathbb{Z}_p$ .*

### 3. Morita's theory for $\mathrm{SL}(2, F)$

In this section, we study Morita's theory for  $\mathrm{Sp}(2, F) = \mathrm{SL}(2, F)$ . We start with reviewing the constructions of holomorphic discrete series and principal series for  $\mathrm{SL}(2, F)$

from [5], [6] and [7] in accordance with our notations. Then we focus on the duality established in §2 for  $\mathrm{SL}(2, F)$  and its relation with Morita's duality and Casselman's intertwining operator.

**3.1. The  $p$ -adic upper half-plane.** For more details, we refer the readers to [5] §2 and [2] 1.2.

In the following, let  $G = \mathrm{SL}(2, F)$  and  $G_0 = \mathrm{SL}(2, \mathfrak{o})$ .

Let  $\Sigma := K - F$  be the  $p$ -adic upper half-plane,  $\mathcal{P}$  be the set of nonzero pairs  $(x, y) \in F \times F$ ,  $\mathcal{L} = F^\times \backslash \mathcal{P} = \mathbb{P}^1(F)$ ,  $\mathcal{P}_0 \subset \mathcal{P}$  be the set of pairs  $(x, y) \in \mathfrak{o} \times \mathfrak{o}$  such that  $(x, y) \not\equiv (0, 0) \pmod{\varpi}$ . As usual, we define a  $G$ -action on  $\Sigma$  by

$$g \cdot Z := (aZ + b)(cZ + d)^{-1}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G.$$

Let  $m$  be a nonnegative integer. For a pair  $(x, y) \in \mathcal{P}_0$ , we define

$$B^-(m; x, y) := \{Z \in K \mid |xZ + y| < \max\{1, |Z|\} |\varpi^m|\}.$$

Let

$$\begin{aligned} \Sigma(m) &:= \bigcap_{(x, y) \in \mathcal{P}_0} K - B^-(m; x, y) \\ &= \{Z \in \Sigma \mid |xZ + y| \geq \max\{1, |Z|\} |\varpi^m| \text{ for any } (x, y) \in \mathcal{P}_0\}. \end{aligned}$$

It is not hard to verify that the admissible affinoid covering  $\{\Sigma(m)\}_{m=0}^\infty$  of  $\Sigma$  coincide with that defined in [2] 1.2.

Let  $\mathcal{O}(\Sigma(m))$  be the space of  $K$ -valued rigid analytic functions on  $\Sigma(m)$ . Explicitly, on taking partial fractional expansion of each summand in (1.7), one sees that  $\psi \in \mathcal{O}(\Sigma(m))$  is a  $K$ -valued functions on  $\Sigma(m)$  that has an expansion in the form:

$$\psi(Z) = \sum_{i=0}^{\infty} a_i^{(\infty)} Z^i + \sum_{j=1}^{\ell} \sum_{i=-1}^{-\infty} a_i^{(j)} (Z - z_j)^i,$$

where  $\ell \geq 0$ ,  $a_i^{(\infty)}, a_i^{(j)} \in K$ ,  $z_j \in F$ , and the expansion converges with respect to the supremum norm. The space of  $K$ -rigid analytic functions on  $\Sigma$  is the projective limit of  $\mathcal{O}(\Sigma(m))$ .

**3.2. Holomorphic discrete series of  $\mathrm{SL}(2, F)$ .** Let  $s$  be an integer. We define the holomorphic discrete series  $(\mathcal{O}(\Sigma), \pi_s)$  of  $G$  (see (1.10); compare [5] §3-1.):

$$(3.1) \quad \pi_s(g)\psi(Z) := (-cZ + a)^{-s} \psi((dZ - b)(-cZ + a)^{-1}),$$

with  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$  and  $\psi \in \mathcal{O}(\Sigma)$ .  $\pi_s$  is a continuous representation of  $G$ .

Let  $\mathcal{N}_s^0(\Sigma)$  be the subspace of  $\mathcal{O}(\Sigma)$  spanned by 1 and  $\psi_z^{(s)}(Z) := (Z - z)^{-s}$  for all  $z \in F$  (see (2.10)), and let  $\mathcal{N}_s(\Sigma)$  be the closure of  $\mathcal{N}_s^0(\Sigma)$ .  $\mathcal{N}_s(\Sigma)$  is  $G$ -invariant.

If  $s \leq 0$ ,  $\mathcal{N}_s(\Sigma)$  is obviously the space of polynomial functions of degree  $\leq -s$ .

If  $s > 0$ , let  $\mathcal{N}_s^0(\Sigma)$  be the subspace of  $\mathcal{O}(\Sigma)$  consisting of all rational functions  $\psi$  that has a partial fractional expansion of the form

$$\psi(Z) = \sum_{i=0}^{\infty} a_i^{(\infty)} Z^i + \sum_{j=1}^{\ell} \sum_{i=-s}^{-\infty} a_i^{(j)} (Z - z_j)^i,$$

where the sum is finite, with  $\ell \geq 0$ ,  $z_j \in F$  and  $a_i^{(\infty)}, a_i^{(j)} \in K$ . One verifies that  $\mathcal{N}_s^0(\Sigma)$  is  $G$ -invariant. Moreover, let  $\tilde{\mathcal{N}}_s(\Sigma)$  be the closure of  $\mathcal{N}_s^0(\Sigma)$  in  $\mathcal{O}(\Sigma)$ .

The next lemma follows immediately from [5] Theorem 2 (i).

**LEMMA 3.1.** *Let  $s$  be a positive integer.  $\tilde{\mathcal{N}}_s(\Sigma)$  is the smallest  $G$ -invariant closed subspace of  $\mathcal{O}(\Sigma)$  containing 1.*

We note that  $1 \in \mathcal{N}_s(\Sigma)$  and  $\mathcal{N}_s^0(\Sigma) \subset \tilde{\mathcal{N}}_s(\Sigma)$ , and therefore we have the following proposition.

**PROPOSITION 3.2.** *Let  $s$  be a positive integer.  $\mathcal{N}_s(\Sigma) = \tilde{\mathcal{N}}_s(\Sigma)$ .*

**3.3. Principal series of  $SL(2, F)$ .** The references for this section are [6] §2, 3 and [7] §2.

Let  $s$  be an integer. Define the character of  $F^\times$ ,  $\chi_s(z) = z^s$ . Let  $C_{\chi_s}^{\text{an}}(\mathcal{P})$  denote the space of  $K$ -valued locally analytic functions  $\varphi$  on  $\mathcal{P}$  satisfying

$$\varphi(hx, hy) = \chi_s(h)\varphi(x, y), \quad (x, y) \in \mathcal{P}, h \in F^\times.$$

In the following, we identify  $(1, F)$  with  $F$  via  $(1, -z) \rightarrow z$  and write  $\varphi(z) = \varphi(1, -z)$  and  $\varphi(\infty) = \varphi(0, 1)$ . Then  $\varphi(z)$  is a locally analytic function on  $F$  that has Laurent expansion at infinity of the form:

$$\varphi(z) = \sum_{i=s}^{-\infty} b_i^{(\infty)} z^i, \quad b_i^{(\infty)} \in K.$$

Clearly,  $\varphi(\infty) = (-1)^s b_s^{(\infty)}$ . Let  $D_s$  denote the space of all such functions  $\varphi(z)$  on  $F$ . We have a  $K$ -linear bijective map between  $D_s$  and  $C_{\chi_s}^{\text{an}}(\mathcal{P})$ ; we endow  $D_s$  with the topology that makes this map into an isomorphism. Then the representation  $(C_{\chi_s}^{\text{an}}(\mathcal{P}), T_{\chi_s})$  of  $G$ , defined by (1.4), can be realized as the representation  $(D_s, T_s)$ :

$$(3.2) \quad T_s(g)\varphi(z) := (-cz + a)^s \varphi((dz - b)(-cz + a)^{-1}), \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G, \varphi \in D_s.$$

Let  $B_s^0$  be the subspace of  $D_s$  spanned by the  $K$ -valued locally analytic functions  $\varphi_Z^{(s)}(z) := (Z - z)^s$  for all  $Z \in \Sigma$  (see (2.1)), and let  $B_s$  be the closure of  $B_s^0$ .  $B_s$  is  $G$ -invariant.

If  $s$  is a nonnegative integer, let  $P_s^{\text{loc}}$  denote the subspace of  $D_s$  consisting of  $K$ -valued functions  $\varphi(z)$  on  $F$  such that the local Taylor expansion at each point of  $F$  and the Laurent expansion at infinity of  $\varphi(z)$  are both given by polynomials of degree  $\leq s$ .  $P_s^{\text{loc}}$  is  $G$ -invariant. Observe that  $\varphi \in P_s^{\text{loc}}$  if and only if its  $(s+1)$ -th derivation  $(d/dz)^{s+1}\varphi(z) \equiv 0$ .

Let  $P_s$  denote the space of all polynomial functions on  $F$  of degree  $\leq s$ .  $P_s^{\text{loc}}$  and  $P_s$  are both closed  $G$ -invariant subspaces. Clearly  $B_s = P_s$ .

The subspace of  $C_{\chi_s}^{\text{an}}(\mathcal{P})$  corresponding to  $P_s$  (resp.  $P_s^{\text{loc}}$ ) is the space of (resp. locally) homogeneous polynomial functions  $\varphi(x, y)$  on  $\mathcal{P}$  of degree  $s$ .

In addition, we define  $P_{-1}^{\text{loc}} = P_{-1} = 0$ .

**PROPOSITION 3.3** (Casselman's intertwining operator). *Let  $s \geq -1$ . The  $s + 1$ -th differentiation map*

$$(3.3) \quad \begin{aligned} S_s : D_s &\rightarrow D_{-s-2} \\ \varphi(z) &\mapsto (d/dz)^{s+1} \varphi(z) \end{aligned}$$

induces a  $G$ -isomorphism from  $D_s/P_s^{\text{loc}}$  onto  $D_{-s-2}$ .

### 3.4. Morita's duality for $\text{SL}(2, F)$ .

**DEFINITION 3.4** (cf. [6] §5.). *Let  $s$  be an integer.*

1. *We call the following  $K$ -linear pairing  $\langle \cdot, \cdot \rangle_M^{(s)} : D_{s-2} \times \mathcal{O}(\Sigma) \rightarrow K$  **Morita's pairing**:*

$$(3.4) \quad \langle \varphi, \psi \rangle_M^{(s)} := \text{the sum of residues of the 1-form } \varphi(z)\psi(z) dz \text{ on } \mathcal{L},$$

where  $\varphi \in D_{s-2}$  and  $\psi \in \mathcal{O}(\Sigma)$ .

2. *For  $\psi \in \mathcal{O}(\Sigma)$ , let  $M_s(\psi)$  be the linear functional of  $D_{s-2}$  defined by*

$$\langle \varphi, M_s(\psi) \rangle = \langle \varphi, \psi \rangle_M^{(s)}, \quad \varphi \in D_{s-2}.$$

$M_s : \mathcal{O}(\Sigma) \rightarrow (D_{s-2})^*$  is called **Morita's duality operator**.

By some explicit computations of  $\langle \cdot, \cdot \rangle_M^{(s)}$  (ibid.), we obtain the following proposition.

**PROPOSITION 3.5** (Compare ibid. Theorem 3). *Let  $s$  be an integer.*

(1) *If  $s > 0$ , then  $M_s$  induces isomorphisms of  $G$ -representations*

$$(\mathcal{O}(\Sigma), \pi_s) \xrightarrow{\cong} ((D_{s-2}/P_{s-2})_b^*, T_{s-2}^*)$$

and

$$(\mathcal{N}_s(\Sigma), \pi_s) \xrightarrow{\cong} ((D_{s-2}/P_{s-2}^{\text{loc}})_b^*, T_{s-2}^*).$$

(2) *If  $s \leq 0$ , then  $M_s$  induces isomorphisms of  $G$ -representations*

$$(\mathcal{O}(\Sigma)/\mathcal{N}_s(\Sigma), \pi_s) \xrightarrow{\cong} ((D_{s-2})_b^*, T_{s-2}^*).$$

We still denote the isomorphisms in (1) and (2) by  $M_s$ .

**3.5. The duality operator  $I_s$ .** We define a continuous linear operator  $I_s$  from  $(B_{-s})_b^*$  to  $\mathcal{N}_s(\Sigma)$  (see §2.1)

$$(3.5) \quad I_s(\xi)(Z) := \langle \varphi_Z^{(-s)}, \xi \rangle.$$

THEOREM 3.6. *If  $s$  is a positive integer, then we have a commutative diagram:*

$$\begin{array}{ccc} (\mathcal{N}_s(\Sigma), \pi_s) & \xrightarrow{(s-1)! M_s} & ((D_{s-2}/P_{s-2}^{\text{loc}})^*, T_{s-2}^*) \\ \uparrow I_s & & \uparrow S_{s-2}^* \\ ((B_{-s})_b^*, T_{-s}^*) & \xlongequal{\quad} & ((D_{-s})_b^*, T_{-s}^*) \end{array}$$

PROOF. Let  $i$  be the inclusion:  $B_{-s} \hookrightarrow D_{-s}$ . Then  $i^* : (D_{-s})_b^* \rightarrow (B_{-s})_b^*$  is surjective due to the Hahn-Banach Theorem. According to Theorem 2.6, Proposition 3.5 (1) and Proposition 3.3, the maps  $I_s$ ,  $(s-1)! M_s$  and  $S_{s-2}^*$  in the diagram are all isomorphisms of  $G$ -representations. Therefore it suffices to prove the commutativity of the following diagram:

$$\begin{array}{ccc} (\mathcal{N}_s(\Sigma), \pi_s) & \xrightarrow{(s-1)! M_s} & ((D_{s-2}/P_{s-2}^{\text{loc}})^*, T_{s-2}^*) \\ \swarrow I_s & & \nwarrow i^* \circ (S_{s-2}^{-1})^* \\ & ((B_{-s})_b^*, T_{-s}^*) & \end{array}$$

We define  $\xi_\infty \in (B_{-s})^*$  by  $\langle \varphi_Z^{(-s)}, \xi_\infty \rangle = \varphi_Z^{(-s)}(\infty) = 1$ , then  $I_s(\xi_\infty)(Z) = 1$  by definition (3.5). Since  $\pi_s(g)1$ , for all  $g \in G$ , topologically spans  $\mathcal{N}_s$ , we are reduced to proving

$$(3.6) \quad (s-1)! (S_{s-2}^{-1})^* \circ M_s(1) = \xi_\infty.$$

For any  $Z \in \Sigma$ , we have  $S_{s-2}(\varphi_Z^{(-1)}) = (s-1)! \varphi_Z^{(-s)}$ , hence

$$\begin{aligned} & \langle \varphi_Z^{(-s)}, (s-1)! (S_{s-2}^{-1})^* \circ M_s(1) \rangle \\ &= \langle (s-1)! S_{s-2}^{-1}(\varphi_Z^{(-s)}), M_s(1) \rangle \\ &= \langle \varphi_Z^{(-1)}, 1 \rangle_M^{(s)} \\ &= \text{Res}_\infty(Z - z)^{-1} dz \\ &= 1 \\ &= \langle \varphi_Z^{(-s)}, \xi_\infty \rangle. \end{aligned}$$

Since  $\varphi_Z^{(-s)}$ , for all  $Z \in \Sigma$ , topologically spans  $B_{-s}$ , (3.6) follows. Q.E.D.

If  $s \leq 0$ , then  $I_s : (B_{-s})_b^* \rightarrow \mathcal{N}_s(\Sigma)$  is an isomorphism between two  $(-s+1)$ -dimensional  $G$ -representations.

#### 4. Concluding remarks

Professor P. Schneider pointed out that the  $p$ -adic Siegel upper half-space  $\Sigma$  was constructed in M. van der Put and H. Voskuil's paper [13] as the symmetric space associated to the symplectic group  $G = \text{Sp}(2n, F)$ . In fact, if we let  $P^-$  denote the transpose of  $P$ , and  $G$ ,  $U$  and  $P^-$  the  $F$ -rigid analytifications of  $G$ ,  $U$  and  $P^-$ , respectively, then  $\Sigma$  can be realized as the complement of all the  $G$ -translations of  $(G - U \cdot P^-)/P^-$  in  $G/P^-$ . However, the construction of the affinoid covering using the Bruhat-Tits building in [13] is different from ours.

We claim that this observation enables us to generalize most of the constructions and results in this article to split reductive groups.

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